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A tool for the Analytic Hierarchy Process based on Leibniz's formula for determinants computation

Uma ferramenta para a Análise Hierárquica de Processos baseada na fórmula de Leibniz para cálculo do determinante

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Abstract

In the Analytic Hierarchy Process (AHP) are expected consistent matrices when the decision-maker performs perfect judgements. Such matrices rarely appear in real situations, since humans do not always decide in the same way. However, understanding these matrices helps to understand the near consistent matrices, which actually occur in real situations. Therefore, in this paper the Leibniz's formula properties are explored for these matrices determinant calculus. The main objective of this approach is to insert, in the AHP framework, new theories to the study of consistent and near consistent matrices. For that, two models of near consistent matrices are implemented, namely multiplicative and additive. In addition, some consequent results are explored, such as, diagonalization and exponential matrix.

Keywords: Analytic Hierarchy Process. Decision-making. Consistent and near consistent matrices. Leibniz's formula. Determinant.

Resumo

Matrizes consistentes são esperadas na Análise Hierárquica de Processos (AHP) quando o tomador de decisão executa julgamentos perfeitos. Tais matrizes raramente aparecem em situações reais, pois humanos nem sempre decidem da mesma forma. No entanto, entender essas matrizes ajuda a entender as matrizes quase consistentes, que realmente ocorrem em situações reais. Portanto, neste artigo são exploradas as propriedades da fórmula de Leibniz para o cálculo do determinante dessas matrizes. O objetivo principal desta abordagem é inserir no contexto do AHP novas teorias para o estudo de matrizes consistentes e quase consistentes. Para tanto, são implementados dois modelos de matrizes quase consistentes, a saber, multiplicativo e aditivo. Além disso, alguns resultados consequentes dessa abordagem são explorados, como a diagonalização de matrizes e a matriz exponencial.

Palavras-chave: Análise Hierárquica de Processos. Tomada de decisão. Matriz consistente e quase-consistente. Fórmula de Leibniz. Determinante.

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1 Introduction

The Analytic Hierarchy Process (AHP) allows modeling a decision problem as a hierarchical structure. The decision-maker, from pairwise comparisons, connects the elements to this structure (SAATY, 1987). The judgements result is a square pairwise comparison matrix $A = (a_{ij})_{n\times n}$, for which each element gives the relative importance of one alternative over another (ALONSO; LAMATA, 2006). This matrix is named positive reciprocal because it has the property that the elements in the main diagonal are unitary, and the elements in the i-th row are the inverse of the elements in the i-th column (SAATY, 2008). If $a_{ij}a_{jk} = a_{ik}$, for $i, j, k = 1, \cdots, n$, then the pairwise comparison matrix is said to be a consistent matrix (SAATY, 2003).

Human judgements must obey a transitive relationship (SAATY, 2008), i.e., if alternative φ_m is preferable over φ_p, and the latter is preferable over φ_q, then φ_m is preferable over alternative φ_q. The consistency (or near consistency) of judgements is very important mainly on situations where the decisions lead to critical results (OLIVEIRA; OLIVEIRA; DUARTE, 2016). Many researchers have addressed the matrices of consistent type. Pelaez and Lamata (2003) define a consistency index based on matrix transitivity. Alonso and Lamata (2006) propose a new consistency analysis approach, which is adaptable for a specific scope. Xu (2000) investigates the consistency of weighted mean complex judgement matrices. Leung and Cao (2000) propose a fuzzy consistency. Aull-Hyde, Erdogan and Duke (2006) perform a research about consistency of aggregated comparison, and Lamata and Pelaez (2002) establish a theorem that the determinant of an order-three consistent matrix is null.

In this paper, we address the theory for calculating determinants based on Leibniz's formula and we present the theory fundamentals aiming to contribute with the AHP framework. In order to demonstrate their use, we highlight two main results: the proof that a consistent matrix determinant is null and that the near consistent matrix determinant is equal to the perturbation matrix determinant. We also present other consequential results.

2 Materials and methods

In this section, the mathematical theory required to understand the proposed approach is established.

Definition 2.1. A square matrix $\boldsymbol{A} = (a_{ij})_{n\times n}, n \geq 2$, is a pairwise comparison positive reciprocal matrix if each element a_{ij} is a judgement of the decision-maker with respect to alternative φ_i over alternative φ_j, where $a_{ii} = 1$, $a_{ij} = 1/a_{ji}$ and $a_{ij} > 0$ for all $i, j = 1, 2, \cdots, n$.

The values $a_{ij} = w_i/w_j$ represent estimations of judgements whose precise values are w_i and w_j. Vector of priorities $\boldsymbol{w} = [w_1\ w_2\ \cdots\ w_n]$ can be obtained by solving the system $\boldsymbol{Aw} = \lambda_{max}\boldsymbol{w}$ (SAATY, 2008), where λ_{max} is the largest eigenvalue from the singular value decomposition of matrix $\boldsymbol{A}$.

When $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1, 2, \cdots, n$, $\boldsymbol{A}$ is said to be a consistent matrix (SAATY, 2003). If the judgements are inconsistent then the resultant matrix will also be inconsistent. An inconsistent judgement means that: the alternative φ_1 is preferable to φ_2, and φ_2 is preferable to φ_3, but φ_1 is not preferable to φ_3. In other words, the transitive property is not applied to inconsistent matrices. It is easy to notice that, if a matrix of judgements is an order-two matrix, then it can only be a consistent matrix.

A near consistent matrix can be obtained by means of two models: multiplicative and additive. On the multiplicative model, an $n \times n$ consistent matrix is multiplied by an $n \times n$ perturbation matrix $\boldsymbol{E}$. This way, $\boldsymbol{A} = \boldsymbol{E} \circ \boldsymbol{W}$, where $\circ$ is the Hadamard product, $a_{ij} = \varepsilon_{ij} w_i / w_j$, $\boldsymbol{W} = (w_i/w_j)_{n\times n}$, $\boldsymbol{E} = (\varepsilon_{ij})_{n\times n}$ and each element in $\boldsymbol{E}$ satisfies $\varepsilon_{ji} = 1/\varepsilon_{ij}$, with small values, close to one (SAATY, 2003). If $\boldsymbol{E}$ is the identity matrix under the Hadamard product, then $\boldsymbol{A}$ is a consistent matrix.

In the additive model, the near consistent matrix is the outcome of additive perturbations considering that $a_{ij} = w_i/w_j + \gamma_{ij}$ (SAATY, 2003). It results in $\boldsymbol{A} = \boldsymbol{\Gamma} + \boldsymbol{W}$, where $\boldsymbol{\Gamma} = (\gamma_{ij})_{n\times n}$ and the perturbation matrices elements are respectively related by Equations (1) and (2) (SAATY, 2003):

$$\varepsilon_{ij} = 1 + \frac{w_j}{w_i}\gamma_{ij} \text{ and} \tag{1}$$

$$\boldsymbol{\Gamma} = (\boldsymbol{E} - \boldsymbol{J}) \circ \boldsymbol{W}, \tag{2}$$

where $\boldsymbol{J} = (1_{ij})_{n\times n}$ is the identity matrix under the Hadamard product. Note that the main diagonal elements in $\boldsymbol{\Gamma}$ are null since $\boldsymbol{E}$ and $\boldsymbol{W}$ are positive reciprocal matrices, so that $\varepsilon_{ii} = w_i/w_i = 1$ and $\gamma_{ij} = (\varepsilon_{ij} - 1) w_i/w_i$, if $\boldsymbol{A}$ is a consistent matrix.

Example 2.1. Matrix

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 2 & 8 \\ 1/2 & 1 & 4 \\ 1/8 & 1/4 & 1 \end{bmatrix},$$

is a consistent matrix, so $det(\boldsymbol{Q}) = 0$. In order to change its consistency, we choose the perturbation matrix

$$\boldsymbol{E} = \begin{bmatrix} 1 & 13/14 & 51/56 \\ 14/13 & 1 & 7/8 \\ 56/51 & 8/7 & 1 \end{bmatrix}.$$

In this way, a near consistent matrix is obtained by the Hadamard product as $\boldsymbol{A} = \boldsymbol{E} \circ \boldsymbol{Q}$,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 13/7 & 51/7 \\ 7/13 & 1 & 7/2 \\ 7/51 & 2/7 & 1 \end{bmatrix}.$$

One can verify that $det(\boldsymbol{A}) = det(\boldsymbol{E})$. We will prove this general result in the next section for any pairwise comparison reciprocal matrix. Now, we consider the additive model and the elements of the perturbation matrix obtained in Equation (1):

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0 & -1/7 & -5/7 \\ 1/26 & 0 & -1/2 \\ 5/408 & 1/28 & 0 \end{bmatrix}.$$

Therefore, a near consistent matrix is obtained by the addition $\boldsymbol{B} = \boldsymbol{Q} + \boldsymbol{\Gamma}$. It is clear that $\boldsymbol{B} = \boldsymbol{A}$ since, from Equation (2), $\boldsymbol{Q} + \boldsymbol{\Gamma} = \boldsymbol{E} \circ \boldsymbol{Q} = \boldsymbol{A}$. Note that $det(\boldsymbol{\Gamma}) = 0$. We demonstrate this result shortly.

In addition to the AHP theory, a brief study on permutations is necessary, since they are important tools for the development of this work. From them, it will be addressed the Leibniz's formula for determinants computation that is used in the course of theorems and propositions demonstrations.

Let S_n be the set of bijective mappings $\tau: \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$. τ is called a permutation of $\{1, 2, \cdots, n\}$ and S_n is the symmetric group of degree n, where n is a positive integer. Note that there are $n!$ permutations. The τ function has a unique inverse function

$\tau^{-1}: \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$, which is also a permutation. Moreover, if σ and τ are permutations, composite $(\sigma \circ \tau)(i) = \sigma(\tau(i))$ is also a permutation. A convenient way to denote the τ permutation is

$$\tau = \begin{pmatrix} 1 & \cdots & n \\ \tau(1) & \cdots & \tau(n) \end{pmatrix},$$

where the second row consists in the respective images from the elements in the first row under τ. In this case τ is the identity permutation, since $\tau(i) = i, \forall i \in \{1, 2, \cdots, n\}$. Such permutation is expressed as τ_{id}.

An important concept is that the elements of the S_n group are related to the sign of a permutation. The $sgn(\cdot)$ function is necessary to calculate the determinant through the Leibniz's formula. This function computes how much a permutation τ changes the order of its elements in comparison to τ_{id}. Let $\eta(\tau)$ be the number of pairs (p, q) such that $p < q$, but $\tau(p) > \tau(q)$, i.e., (JÄNICH, 1991):

$$\eta(\tau) = \#\{(p, q) |\, p < q \; but \; \tau(p) > \tau(q)\},$$

where # returns the number of pairs (p, q). The sign of a τ permutation, represented by $sgn(\tau)$, is defined as

$$sgn(\tau) = \begin{cases} 1, \text{ if } \eta(\tau) \text{ is even} \\ -1, \text{ if } \eta(\tau) \text{ is odd} \end{cases}.$$

Therefore, $sgn(\tau)$ depends only on the parity of τ. Two important properties are $sgn(\tau \cdot \sigma) = sgn(\tau)sgn(\sigma)$ and $sgn(\tau^{-1}) = sgn(\tau)$. The calculus of the permutation τ sign is made by Equation (3) (ROBINSON, 2003):

$$sgn(\tau) = (-1)^{\eta(\tau)}. \quad (3)$$

Example 2.2. Let $S_3 = \{1, 2, 3\}$ be a group. We consider the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

that is $\tau(1) = 3$, $\tau(2) = 2$ and $\tau(3) = 1$. Therefore,

$$1 < 2 \text{ but } \tau(2) < \tau(1)$$
$$1 < 3 \text{ but } \tau(3) < \tau(1).$$
$$2 < 3 \text{ but } \tau(3) < \tau(2)$$

Since $\eta(\tau) = 3$ then, according to Equation (3), $sgn(\tau) = -1$.

Given $\tau \in S_n$, the support of τ is defined to be the set of all i such that $\tau(i) \neq 1$, symbolically $supp(\tau)$. A permutation τ is called an r-cycle if $supp(\tau) = \{i_1, i_2, \cdots, i_r\}$ with distinct i_k, where $\tau(i_1) = i_2$, $\tau(i_2) = i_3$, $\cdots$, $\tau(i_{r-1}) = i_r$ and $\tau(i_r) = i_1$. For this reason, τ is often written in the form $\tau = (i_1 \, i_2 \cdots \, i_r)(i_{r+1}) \cdots (i_n)$. A permutation that does not switch more than two adjacent numbers and leaves the remaining $n - 2$ fixed is a neighbour transposition or, simply, transposition. This way a transposition is a 2-cycle. Hence, if σ is a transposition, then $\eta(\sigma \cdot \tau) = \eta(\tau) \pm 1$.

In the following, we present two results that will be useful throughout the text, especially in Theorem 2.1. In Robinson (2003), we found proofs and details of these results.

Proposition 2.1. If τ is a transposition then $sgn(\tau) = -1$.

Proposition 2.2. A permutation $\tau \in S_n$ is even (odd) if and only if it is a product of an even (odd) number of transpositions.

Theorem 2.1, establishes an auxiliary result to demonstrate the theorems about determinants of consistent and near consistent matrices. We show its proof, since it is very important to the text. In the proof, symbol $(1\;2)$ means that first column of permutation τ is written in the form of Example 2.2.

Theorem 2.1. For $n > 1$, there are $\frac{1}{2}n!$ even permutations and $\frac{1}{2}n!$ odd permutations in S_n.

Proof. Let A_n be the set of all even permutations in S_n. Defining a function $\alpha: A_n \to S_n$ by the rule $\alpha(\tau) = \tau \cdot (1\ 2)$, observing that $\alpha(\tau)$ is odd and α is injective, due to Propositions 2.1 and 2.2, one has

$$sgn\big(\tau \cdot (1\ 2)\big) = sgn(\tau)sgn\big((1\ 2)\big) = 1(-1) = -1, \text{ and}$$
$$\tau_1 \cdot (1\ 2) = \tau_2 \cdot (1\ 2) \Longrightarrow \tau_1 \cdot (1\ 2) \cdot (1\ 2) = \tau_2 \cdot (1\ 2) \cdot (1\ 2) \Longrightarrow$$
$$\tau_1 = \tau_2.$$

Every odd permutation σ belongs to the image set $Im(\alpha)$, since $\alpha(\tau) = \sigma$, where $\tau = \sigma \cdot (1\ 2) \in A_n$. Therefore, $Im(\alpha)$ is precisely the set of all odd permutations and $|Im(\alpha)| = |A_n|$ where $|X|$ means the cardinality of X.□

Shiraishi, Obata and Daigo (1998), give a last proposition in order to state a basis of results for what follows in next section, which is related to a consistent matrix and its characteristic polynomial.

Proposition 2.3. $\boldsymbol{A} = (a_{ij})_{n\times n}$ is a consistent matrix if and only if $p_{\boldsymbol{A}}(\lambda) = \lambda^n - n\lambda^{n-1}$, where $p_{\boldsymbol{A}}(\lambda)$ is the characteristic polynomial of $\boldsymbol{A}$.

3 Results and discussion

Leibniz's formula provides a means to obtain the determinant in terms of the matrix elements permutations using the results presented in Section 2. The Leibniz's formula is given by Equation (4) for a matrix $\boldsymbol{A} = (a_{ij})_{n\times n}$

$$det(\boldsymbol{A}) = \sum_{\tau\in S_n} sgn(\tau)\, a_{i\tau(1)} \cdots a_{n\tau(n)}, \tag{4}$$

where τ is a permutation in the S_n group. Jänich (1991) show more details.

Considering a new approach proposition, some changes in the notations of τ permutations are considered. We rewrite Equation (4), using Equation (3), as

$$det(\boldsymbol{A}) = \sum_{t=1}^{n!}\left[(-1)^{\eta(\tau_t)} \prod_{i=1}^{n} a_{i\tau_t(i)}\right], \tag{5}$$

where $\tau_t(i)$ is t-th permutation in the S_n group. Next, we present a lemma for fixing a useful result necessary in the main theorems proofs.

Lemma 3.1. Let $\tau \in S_n$ be a permutation from the set $Q = \{1, 2, \cdots, n\}$ and $\boldsymbol{A} = (a_{ij})_{n\times n}$ be a matrix. For all $t, j \in Q$, $\prod_{i=1}^{n} \frac{a_{ij}}{a_{\tau_t(i)j}} = 1$.

Proof. For any $t \in Q$ there is always an element $a_{\tau_t(i)j} = a_{qj}$, where $q \in Q$. Hence,

$$\prod_{i=1}^{n} \frac{a_{ij}}{a_{\tau_t(i)j}} = \frac{a_{1j}a_{2j}\cdots a_{nj}}{a_{\tau_t(1)j}a_{\tau_t(2)j}\cdots a_{\tau_t(n)j}} = \frac{a_{1j}a_{2j}\cdots a_{nj}}{a_{1j}a_{2j}\cdots a_{nj}} = 1. \ \square$$

At last, we state the main results of this work based on the previously established. The following theorem uses Leibniz's formula, consistent matrices properties, permutations characteristics according to Lemma 3.1 and the result provided by Theorem 2.1. From now on, we refer the matrix $\boldsymbol{A} = (a_{ij})_{n\times n}$ as a pairwise comparison positive reciprocal matrix.

Theorem 3.1. If $\boldsymbol{A}$ is a consistent matrix, then $det(\boldsymbol{A}) = 0$.

Proof. Since $\boldsymbol{A}$ is a consistent matrix, then $a_{ij}a_{jk} = a_{ik}$. Setting $k = \tau_u(i)$ then $a_{ij}a_{j\tau_u(i)} = a_{i\tau_u(i)}$, where $\tau_u(i) \in S_n$ is a permutation, $u = 1, 2, \cdots, n!$ and $i, j, \tau_u(i) = 1, 2, \cdots, n$. Since $\boldsymbol{A}$ is reciprocal, then $a_{j\tau_u(i)} = 1/a_{\tau_u(i)j}$, so $a_{ij}/a_{\tau_u(i)j} = a_{i\tau_u(i)}$. Overriding this result in Equation (5), it follows that

$$det(\boldsymbol{A}) = \sum_{u=1}^{n!}\left[(-1)^{\eta(\tau_u)}\prod_{i=1}^{n}\frac{a_{ij}}{a_{\tau_u(i)j}}\right].$$

Due to the result established in Lemma 3.1, one has $\prod_{i=1}^{n}\frac{a_{ij}}{a_{\tau_u(i)j}} = 1$. Therefore,

$$det(\boldsymbol{A}) = \sum_{u=1}^{n!}(-1)^{\eta(\tau_u)}.$$

According to Theorem 2.1, there will be $\frac{1}{2}n!$ even values and $\frac{1}{2}n!$ odd values of $\eta(\tau_u)$. Finally, the summation is

$$det(\boldsymbol{A}) = \sum_{u=1}^{n!}(-1)^{\eta(\tau_u)} = \underbrace{(-1)+(-1)+\cdots+(-1)}_{\frac{1}{2}n!\ times} + \underbrace{1+1+\cdots+1}_{\frac{1}{2}n!\ times} = 0.\ \square$$

An immediate consequence from Theorem 3.1 is related to the largest eigenvalue λ_{max}. Proposition 3.1 shows that it is the only non-null eigenvalue from a consistent matrix. This result is already known, but here a new way of demonstrating it is presented.

Proposition 3.1. If $\boldsymbol{A}$ is a consistent matrix, then all its eigenvalues are null, except the maximum that is equal to n, named λ_{max}.

Proof. Since $\boldsymbol{A}$ is a consistent matrix then from Theorem 3.1 $det(\boldsymbol{A}) = 0$. However, $det(\boldsymbol{A}) = \lambda_1\lambda_2\cdots\lambda_n$, so $\lambda_i = 0$ is an eigenvalue of $\boldsymbol{A}$, for some $i = 1,\cdots,n$. Consider the linear system $\boldsymbol{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$, where $\mathbf{v}_i$ is i-th eigenvector of $\boldsymbol{A}$. By definition $\mathbf{v}_i \neq 0$ for all i, consequently there must be some $n \geq \lambda_k \neq 0, k \neq i$, since $n = trace(\boldsymbol{A}) = \sum_{j=1}^{n}\lambda_j$. Therefore

$$n - \lambda_k = \sum_{\substack{j=1 \\ j\neq i,k}}^{n}\lambda_j.$$

From Proposition 2.3, $p_{\boldsymbol{A}}(\lambda) = \lambda^n - n\lambda^{n-1}$. Its zeros are found making $p_{\boldsymbol{A}}(\lambda) = 0$ for some eigenvalue λ, hence

$$p_{\boldsymbol{A}}(\lambda_k) = 0 \Longrightarrow \lambda_k^n - n\lambda_k^{n-1} = 0.$$

Therefore, $\lambda_k = n$ and

$$\sum_{\substack{j=1 \\ j\neq i,k}}^{n}\lambda_j = 0,$$

where λ_k is the largest eigenvalue of $\boldsymbol{A}$, named λ_{max}. □

From Leibniz's formula, we also derive the determinant for a near consistent matrix. Theorem 3.2 states such result.

Theorem 3.2. Let $\boldsymbol{E} = (\varepsilon_{ij})_{n\times n}$ be a perturbation matrix and $\boldsymbol{W} = (w_i/w_j)_{n\times n}$. If $\boldsymbol{A} = \boldsymbol{E}\circ\boldsymbol{W}$ is a near consistent matrix with respect to multiplicative model, then $det(\boldsymbol{A}) = det\ (\boldsymbol{E})$.

Proof. Since $a_{ij} = \varepsilon_{ij}w_i/w_j$, setting $j = \tau_u(i)$, $u = 1,2,\cdots,n!$, $i,j = 1,2,\cdots,n$ and considering Equation (5), one has

$$det(\boldsymbol{A}) = \sum_{u=1}^{n!}\left[(-1)^{\eta(\tau_u)}\prod_{i=1}^{n}\varepsilon_{i\tau_u(i)}\frac{w_i}{w_{\tau_u(i)}}\right].$$

The denominator $w_{\tau_u(i)}$ will always be the product $w_1w_2\cdots w_n$ just changing the terms positions for any u. Therefore, $\prod_{i=1}^{n}w_{\tau_u(i)} = \prod_{i=1}^{n}w_i$ for any u, so

$$\prod_{i=1}^{n}\frac{w_i}{w_{\tau_u(i)}} = \frac{\prod_{i=1}^{n}w_i}{\prod_{i=1}^{n}w_i} = 1,$$

and then, from Leibniz's formula,

$$det(\boldsymbol{A}) = \sum_{u=1}^{n!}\left[(-1)^{\eta(\tau_u)}\prod_{i=1}^{n}\varepsilon_{i\tau_u(i)}\right] = det\ (\boldsymbol{E}).\ \square$$

Note that if $\boldsymbol{E} = \boldsymbol{J}$ (which characterizes a neutral perturbation matrix) then $\boldsymbol{A} = \boldsymbol{E}\circ\boldsymbol{W}$ is a consistent matrix, since $det(\boldsymbol{E}) = 0$. It is still in accordance with Theorem 3.2. We highlight that we obtain the same result taking into account the similarity between $\boldsymbol{A}$ and $\boldsymbol{E}$ (see Proposition 3.2 below).

Considering now the perturbation additive model, Theorem 3.3 provides the determinant only for the related matrix $\boldsymbol{\Gamma}$, because it is clear that the determinant of a near consistent matrix is the same for both perturbation models, owing to relation $\boldsymbol{W} + \boldsymbol{\Gamma} = \boldsymbol{A} = \boldsymbol{E} \circ \boldsymbol{W}$, since $\prod_{i=1}^{n} w_i / w_{\tau_u(i)} = \prod_{i=1}^{n} w_i \,/ \prod_{i=1}^{n} w_i = 1$ (Theorem 3.2 demonstration), from Leibniz's formula, it results that $det(\boldsymbol{W}) = 0$.

Theorem 3.3. Let $\boldsymbol{\Gamma} = (\gamma_{ij})_{n\times n}$ be a perturbation matrix and $\boldsymbol{W} = (w_i/w_j)_{n\times n}$. If $\boldsymbol{A} = \boldsymbol{\Gamma} + \boldsymbol{W}$ is a near consistent matrix with respect to additive model, then $det(\boldsymbol{\Gamma}) = 0$.

Proof. Applying the determinant operator to both sides of the matrix equation $\boldsymbol{A} - \boldsymbol{W} = \boldsymbol{\Gamma}$ results $det(\boldsymbol{A} - \boldsymbol{W}) = det\,(\boldsymbol{\Gamma})$. Since that the elements of $\boldsymbol{A} - \boldsymbol{W}$ are $a_{ij} - w_i/w_j = \gamma_{ij}$, from Equation (5) it follows that

$$det(\boldsymbol{\Gamma}) = \sum_{t=1}^{n!}\left[(-1)^{\eta(\tau_t)} \prod_{i=1}^{n} \gamma_{\tau_t(i)j}\right].$$

When $\tau_t(i) = i$ then $a_{i\tau_t(i)} = 1 = w_i/w_{\tau_t(i)}$, for any t. So, $\prod_{i=1}^{n}\left(a_{i\tau_t(i)} - \frac{w_i}{w_{\tau_t(i)}}\right) = \prod_{i=1}^{n} \gamma_{\tau_t(i)j} = 0$, therefore $det(\boldsymbol{\Gamma}) = 0$. □

Proposition 3.2. Let $\boldsymbol{A} = \boldsymbol{E} \circ \boldsymbol{W} = (a_{ij})_{n\times n}$ and $\boldsymbol{A}' = \boldsymbol{\Gamma} + \boldsymbol{W} = (a'_{ij})_{n\times n}$ be near consistent matrices, with respect to multiplicative and additive models, respectively. Let $\boldsymbol{E} = (\varepsilon_{ij})_{n\times n}$ and $\boldsymbol{\Gamma} = (\gamma_{ij})_{n\times n}$ be perturbation matrices and $\boldsymbol{W} = (w_i/w_j)_{n\times n}$. For these perturbation models $\boldsymbol{A}$ is similar to $\boldsymbol{E}$ and to $\boldsymbol{\Gamma} + \boldsymbol{W}$.

Proof. From Theorem 3.2 we know that $det(\boldsymbol{A}) = det(\boldsymbol{E})$. Let $\boldsymbol{D} = diag(w_1, \cdots, w_n)$ be a diagonal matrix. We can write $det(\boldsymbol{A})\, det(\boldsymbol{D}) = det(\boldsymbol{D})\, det(\boldsymbol{E})$, so $\boldsymbol{AD} = \boldsymbol{DE} \Longrightarrow \boldsymbol{A} = \boldsymbol{DED}^{-1}$. Since $det\,(\boldsymbol{A}) = det(\boldsymbol{\Gamma} + \boldsymbol{W})$, in the same way, $\boldsymbol{A} = \boldsymbol{D}(\boldsymbol{\Gamma} + \boldsymbol{W})\boldsymbol{D}^{-1}$. □

An immediate consequence from Proposition 3.2 is that: $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{\Gamma} + \boldsymbol{W}$ have the same determinant, eigenvalues, rank and characteristic polynomial and $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{E}$ and $\boldsymbol{\Gamma} + \boldsymbol{W}$ are invertible (ROBINSON, 2003). This is in accordance with the above results on consistent and quasi-consistent matrix determinants.

It was presented in Proposition 3.1 that $\lambda = 0$ is an eigenvalue from a consistent matrix $\boldsymbol{A}$. Thus, we can consider the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$:

$$\begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 1/a_{12} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $det(\boldsymbol{A}) = 0$, by Theorem 3.1, the system has infinitely many solutions. The $n - 1$ last equations are multiple of the first equation. Hence,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -(a_{12}x_2 + \cdots + a_{1n}x_n) \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

So, the eigenvector space associated to $\lambda = 0$ is generated by

$$\{(-a_{12}, 1, 0, \dots, 0), (-a_{13}, 0, 1, \dots, 0), \dots, (-a_{1n}, 1, 0, \dots, 1)\},$$

Consider vector $\boldsymbol{v} = [1, a_{12}^{-1}, \dots, a_{1n}^{-1}]$ and from Proposition 3.1, which provides $\lambda_{max} = n$, one can conclude that $\boldsymbol{Av} = n\boldsymbol{v}$. In fact

$$\begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 1/a_{12} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_{12}^{-1} \\ \vdots \\ a_{1n}^{-1} \end{bmatrix} - n \begin{bmatrix} 1 \\ a_{12}^{-1} \\ \vdots \\ a_{1n}^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proposition 3.3. Every consistent matrix $\boldsymbol{A}$ is diagonalizable.

Proof. Consider the n eigenvectors from $\boldsymbol{A}$: $\boldsymbol{v}_1 = [-a_{12}, 1, 0, \dots, 0], \boldsymbol{v}_2 = [-a_{13}, 0, 1, \dots, 0], \dots, \boldsymbol{v}_{n-1} = [-a_{1n}, 1, 0, \dots, 1]$ and $\boldsymbol{v}_n = [1, a_{12}^{-1}, \dots, a_{1n}^{-1}]$. These n eigenvectors are linearly independent, hence $\boldsymbol{A}$ is diagonalizable. □

The result presented in Proposition 3.3 implies that $\boldsymbol{A}$ is similar to the diagonal matrix $\boldsymbol{B} = diag(0,0, \cdots, n)$ which is formed by eigenvalues of $\boldsymbol{A}$, i.e., $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$, where

$$\boldsymbol{P} = \begin{bmatrix} -a_{12} & -a_{13} & -a_{14} & \cdots & -a_{1n} & 1 \\ 1 & 0 & 0 & \cdots & 0 & a_{12}^{-1} \\ 0 & 1 & 0 & \cdots & 0 & a_{13}^{-1} \\ 0 & 0 & 1 & \cdots & 0 & a_{14}^{-1} \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{1n}^{-1} \end{bmatrix} \text{ and}$$

$$\boldsymbol{P}^{-1} = \frac{1}{n} \begin{bmatrix} 1/a_{12} & 1-n & a_{13}/a_{12} & a_{14}/a_{12} & \cdots & a_{1n}/a_{12} \\ 1/a_{13} & a_{12}/a_{13} & 1-n & a_{14}/a_{13} & \cdots & a_{1n}/a_{13} \\ 1/a_{14} & a_{12}/a_{14} & a_{13}/a_{14} & 1-n & \cdots & a_{1n}/a_{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & a_{12}/a_{1n} & a_{13}/a_{1n} & a_{14}/a_{1n} & \cdots & 1-n \\ -1 & -a_{12} & -a_{13} & -a_{14} & \cdots & -a_{1n} \end{bmatrix}.$$

Therefore, due to the similarity between $\boldsymbol{A}$ and $\boldsymbol{B}$, $det(\boldsymbol{A}) = det(\boldsymbol{B}) = 0$. □

Proposition 3.4. The k-th power of a consistent matrix $\boldsymbol{A}$, denoted by $\boldsymbol{A}^k$, is similar to matrix $\boldsymbol{B}^{\boldsymbol{k}} = diag(0,0, \cdots, n^k)$.

Proof. Since $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$, where $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ and $\boldsymbol{B}$ are matrices defined above, one has $\boldsymbol{A}^k = \boldsymbol{PB}^k\boldsymbol{P}^{-1}$. □

Proposition 3.5. The exponential of a consistent matrix $\boldsymbol{A}$, denoted by $e^{\boldsymbol{A}}$, is similar to matrix $e^{\boldsymbol{B}} = diag(1,1, \cdots, e^n)$.

Proof. Observe that $e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \boldsymbol{A}^k/k! = \sum_{k=0}^{\infty} \boldsymbol{PB}^k\boldsymbol{P}^{-1}/k! = \boldsymbol{P} \sum_{k=0}^{\infty} \boldsymbol{B}^k/k!\, \boldsymbol{P}^{-1} = \boldsymbol{P}e^{\boldsymbol{B}}\boldsymbol{P}^{-1}$. Since $e^{\boldsymbol{B}} = diag(e^0, e^0, \cdots, e^n) = diag(1,1, \cdots, e^n)$, it follows the result. □

Proposition 3.6. The determinant of an exponential consistent matrix $e^{\boldsymbol{A}}$ is equal to e^n.

Proof. According to the proof of Proposition 3.5 $det\,(e^{\boldsymbol{A}}) = det(\boldsymbol{P}e^{\boldsymbol{B}}\boldsymbol{P}^{-1}) = det(\boldsymbol{P})det(e^{\boldsymbol{B}})det(\boldsymbol{P}^{-1}) = det(e^{\boldsymbol{B}})$ where $e^{\boldsymbol{B}} = diag(1,1, \cdots, e^n)$ and $\boldsymbol{B} = diag(0,0, \cdots, n) = (b_{ij})_{n \times n}$. However, from Leibniz's formula

$$det(e^{\boldsymbol{B}}) = \sum_{t=1}^{n!} \left[(-1)^{\eta(\tau_t)} \prod_{i=1}^{n} e^{b_{i\tau_t(i)}} \right] = \prod_{i=1}^{n} e^{b_{ii}},$$

since $\tau_t(i) \neq 0$ if $\tau_t(i) = i$ and $\eta(\tau_t)$ is even in this case. Nevertheless, the elements from the diagonal exponential matrix assume only the values

$$e^{b_{ii}} = \begin{cases} e^n, \text{if } i = n \\ 1 \text{, otherwise} \end{cases}.$$

So, $det\ (e^{\boldsymbol{A}}) = e^{n} = e^{trace(\boldsymbol{B})}$. □

4 Conclusion

This theoretical paper addressed a tool based on Leibniz's formula for calculating determinants aiming to complement the framework of AHP theory. It was studied the determinant of consistent and near consistent matrices, present in multi-criteria decision-making processes. Results of this approach, such as diagonalization and exponential consistent matrix, were also explored. Future works include the proposition of new consistent indexes based on Leibniz's formula.

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